SPATIAL LOCALIZATION OF TRANSITIONAL LAYERS IN PROBLEMS OF THE NONLINEAR THEORY OF THERMAL CONDUCTIVITY

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It is shown that, based on solutions for a type of temperature waves, solutions can be constructed which describe spatially localized transitional temperature layers.

Heat-transfer processes in media with a constant thermal conductivity and with the volumetric evolution or absorption of heat can be described by solutions for a type of thermal waves, a characteristic feature of which is the presence of a surface of weak discontinuity, separating regions with a pressure gradient equal to zero and differing from zero. In the particular case of absorbing media, this was pointed out in [1, 2].

In the present work a study has been made of the possibility, in principle, of the existence of spatially localized transitional temperature layers with grad $T \neq 0$, bounded by two surfaces of a weak discontinuity, outside of which $T = \text{const}$. The discussion involves solutions for a type of thermal waves in a medium, in which heat sources or sinks may act.

Let us determine the steady-state distribution of the temperature $T(z)$ in the half-space $z > 0$, filled with the above-mentioned medium. If, at the plane $z = 0$, the value of the temperature is maintained constant, $T(0) = T_w = \text{const}$, and $T(\infty) = T_0 = \text{const}$, the functions

$$T(z) = \begin{cases} T_0 - \dfrac{\gamma}{2a}(z - \zeta_0)^2 & \text{with} \quad 0 \leqslant z \leqslant \zeta_0 \\ T_0 & \text{with} \quad \zeta_0 \leqslant z < \infty \end{cases} \tag{1}$$

$$\zeta_0 = \left[\frac{2a}{\gamma}(T_0 - T_w)\right]^{1/2}$$

are the solution of the problem

$$a\frac{d^2T}{dz^2} + \gamma\theta(|T - T_0|) = 0, \qquad \theta(\alpha = 0) = 0, \qquad \theta(\alpha > 0) = 1$$

$$T(0) = T_w, \qquad T(\infty) = T_0, \qquad \frac{dT}{dz}(\infty) = 0$$

$$T(\zeta_0 - 0) = T(\zeta_0 + 0), \qquad \frac{dT}{dz}(\zeta_0 - 0) = \frac{dT}{dz}(\zeta_0 + 0)$$

and describe the sought temperature distribution. Here ζ_0 is the position of the fixed front of the thermal wave, $T(\zeta_0) = T_0$, $dT/dz(\zeta_0) = 0$; a is the coefficient of thermal diffusivity; the constant coefficient γ is positive in the case of heat sources and negative in the case of heat sinks, whose power is determined by the value of $|\gamma|$. With $\gamma > 0$, $T_w \leq T(z) \leq T_0$; with $\gamma < 0$, $T_0 \leq T(z) \leq T_w$.

If the temperature of the surface $z = 0$ does not remain constant, the temperature distribution must be determined by the solution of the following unsteady-state problem:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial z^2} - \gamma\theta(|T - T_0|)$$

$$T(z, 0) = T_0, \quad T(0, t) = T_w(t), \quad T(\infty, t) = T_0, \quad \frac{\partial T}{\partial z}(\infty, t) = 0$$

$$T[\zeta(t) - 0, t] = T(\zeta(t) + 0, t], \quad \frac{\partial T}{\partial z}[\zeta(t) - 0, t] = \frac{\partial T}{\partial z}[\zeta(t) + 0, t]$$

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where $z = \xi(t)$ is the position of the moving front of the thermal wave.

Specifically, if

$$T_w(t) = T_0 + \frac{a\gamma}{v^2}\left(1 + \frac{v^2}{a}t - \exp\frac{v^2}{a}t\right), \quad v = \text{const} > 0 \tag{2}$$

the temperature distribution in a medium with $z \geq 0$ has the form

$$T(z, t) = \begin{cases} T_0 + \frac{a\gamma}{v^2}\left[1 - \exp - \frac{v}{a}(z - vt)\right] - \frac{\gamma}{v}(z - vt) & \text{with } 0 \leqslant z \leqslant \zeta(t) \\ T_0 & \text{with } \zeta(t) \leqslant z < \infty \end{cases} \tag{3}$$

$$\zeta(t) = vt$$

In this, the front of the thermal wave $z = \xi(t)$ is moving in the medium at the velocity v, while, behind the front, with $\xi(t) \leq z < \infty$, there is an unperturbed medium with a temperature $T(z) \equiv T_0$. We note that, with $\gamma > 0$, at the moment of time $t = t_*$, $t_w(t_*) = 0$, then, with $\gamma > 0$, expressions (2) and (3) are meaningful only with $0 \leq t \leq t_*$.

An analysis of solutions for a type of thermal waves, (1) or (3), permits the conclusion that, when zones with heat sinks and sources exist simultaneously in the medium, the change in the temperature can be completely localized in a transitional layer of finite thickness, outside of which the temperature is constant.

For example, in a medium filling the infinite space $-\infty < z < \infty$, with the limiting conditions $T(-\infty) = T_{01} = \text{const}$, $T(+\infty) = T_{02} = \text{const}$, when $T_{02} > T_{01}$, the following steady-state temperature distribution is indicated:

$$T(z) = \begin{cases} T_{01} & \text{with } -\infty < z \leqslant \zeta_1 \\ T_{01} - \frac{\gamma_1}{2a}(z - \zeta_1)^2 & \text{with } \zeta_1 \leqslant z \leqslant 0 \\ T_{02} - \frac{\gamma_2}{2a}(z - \zeta_2)^2 & \text{with } 0 \leqslant z \leqslant \zeta_2 \\ T_{02} & \text{with } \zeta_2 \leqslant z < \infty \end{cases} \tag{4}$$

$$\zeta_1 = -\sqrt{-\frac{2\gamma_2 a}{\gamma_1}\frac{T_{02} - T_{01}}{\gamma_2 - \gamma_1}}, \qquad \zeta_2 = \sqrt{-\frac{2\gamma_1 a}{\gamma_2}\frac{T_{02} - T_{01}}{\gamma_2 - \gamma_1}}$$

Expressions (4) were obtained under the assumption that, in the segment $(\xi_1, 0)$, there act the heat sinks $\gamma_1 = \text{const} < 0$, and, in the segment $(0, \xi_2)$, the heat sources $\gamma_2 = \text{const} > 0$; at the straight line $-\infty < z < \infty$, the function T(z) is everywhere continuous, together with its first derivative dT/dz, being a limiting solution of the problem

$$\begin{gathered} a\frac{d^2T}{dz^2} + F(z, T) = 0 \\ T(-\infty) = T_{01}, \quad T(+\infty) = T_{02}, \quad \frac{dT}{dz}(\pm\infty) = 0 \\ T(-0) = T(+0), \quad \frac{dT}{dz}(-0) = \frac{dT}{dz}(+0) \\ T(\zeta_{1,2} - 0) = T(\zeta_{1,2} + 0), \quad \frac{dT}{dz}(\zeta_{1,2} - 0) = \frac{dT}{dz}(\zeta_{1,2} + 0) \\ F(z, T) = \begin{cases} \gamma_1\theta(T - T_{01}) & \text{with } z < 0 \\ \gamma_2\theta(T_{02} - T) & \text{with } z > 0 \end{cases} \end{gathered} \tag{5}$$

while, at the surfaces $z = 0$ and $z = \xi_1, \xi_2$, the derivatives d^mT/dz^m, $m \geq 2$ may undergo a discontinuity.

If, in addition, it is assumed that the medium is moving with a constant velocity $w = \text{const}$ in a direction perpendicular to the plane $z = 0$, then, in this case the steady-state temperature distribution must be found by solution of the problem

$$\begin{gathered} a\frac{d^2T}{dz^2} - w\frac{dT}{dz} + F(z, T) = 0 \\ T(-\infty) = T_{01}, \quad T(+\infty) = T_{02}, \quad \frac{dT}{dz}(\pm\infty) = 0 \\ T(-0) = T(+0), \quad \frac{dT}{dz}(-0) = \frac{dT}{dz}(+0) \end{gathered} \tag{6}$$

$$T(\zeta_{1,2}-0)=T(\zeta_{1,2}+0),\quad \frac{dT}{dz}(\zeta_{1,2}-0)=\frac{dT}{dz}(\zeta_{1,2}+0)$$

where F(z, T) is determined in accordance with expressions (5). The solution of the problem (5), (6) is written in the form

$$T(z)=\begin{cases} T_{01} & \text{with } -\infty<z\leqslant\zeta_1 \\ T_{01}+\frac{\gamma_1 a}{w^2}\left[1-\exp\frac{w}{a}(z-\zeta_1)\right]+\frac{\gamma_1}{w}(z-\zeta_1) & \text{with } \zeta_1\leqslant z\leqslant 0 \\ T_{02}+\frac{\gamma_2 a}{w^2}\left[1-\exp\frac{w}{a}(z-\zeta_2)\right]+\frac{\gamma_2}{w}(z-\zeta_2) & \text{with } 0\leqslant z\leqslant\zeta_2 \\ T_{02} & \text{with } \zeta_2\leqslant z<\infty \end{cases} \tag{7}$$

here the values of ζ_1 and ζ_2 must be found from the algebraic system

$$\begin{gathered} T_{01}+\frac{\gamma_1 a}{w^2}\left[1-\exp\left(-\frac{w}{a}\zeta_1\right)\right]-\frac{\gamma_1}{w}\zeta_1=T_{02}+\frac{\gamma_2 a}{w^2}\left[1-\exp\left(-\frac{w}{a}\zeta_2\right)\right]-\frac{\gamma_2}{w}\zeta_2 \\ \frac{\gamma_1}{w}\left[1-\exp\left(-\frac{w}{a}\zeta_1\right)\right]=\frac{\gamma_2}{w}\left[1-\left(\exp\left(-\frac{w}{a}\zeta_2\right)\right)\right] \end{gathered} \tag{8}$$

which, with arbitrary values of w, γ_1, and γ_2, must be solved numerically. In analytical form solutions of system (8) can be obtained with $w=0$; in this case expressions (7) are accordingly written in the form of expressions (4). In addition, system (8) has an analytical expression for the solutions with $-\gamma_1=\gamma_2=\gamma$

$$\zeta_2=-\zeta_1+w\gamma^{-1}(T_{02}-T_{01})$$

$$\zeta_1=-\frac{a}{w}\ln\left\{1+\sqrt{1-\exp\left[-\frac{w^2}{a\gamma}(T_{02}-T_{01})\right]}\right\}\quad\text{with}\quad w>0$$

$$\zeta_1=-\frac{a}{w}\ln\left\{1-\sqrt{1-\exp\left[-\frac{w^2}{a\gamma}(T_{02}-T_{01})\right]}\right\}\quad\text{with}\quad w<0$$

Thus, the presence of heat sources and sinks in the medium, depending nonlinearly on the temperature, can ensure the existence of spatially localized transitional thermal layers. In the cases under consideration, their thickness tends to zero if $T_{02}-T_{01}\to 0$ or if $-\gamma_1, \gamma_2\to\infty$. If the difference $T_{02}-T_{01}$ remains a finite quantity, and $-\gamma_1, \gamma_2\to\infty$, then, on the curve of the temperature distribution there is a discontinuity, i.e., a temperature "offset."

We note that transitional temperature layers may also be observed in a consideration of media with heat sources and sinks of a more general form than are discussed in the present paper, and with another arrangement of the sources and sinks, for example, if they are far apart. The investigation of the structure of transitional temperature waves and their displacements in space is of interest in the study of high-temperature hydrodynamic phenomena, accompanied by radiation and the absorption of light quanta [3]. However, in this case the corresponding analytical investigations are made considerably more difficult by the complex character of the action of the heat sources and sinks, as well as by their complex interconnection.

LITERATURE CITED

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